Positive Feedback, Relaxation Oscillators and Chaos*

Lecture 2, Part 2 – Relaxation Oscillators and Chaos*

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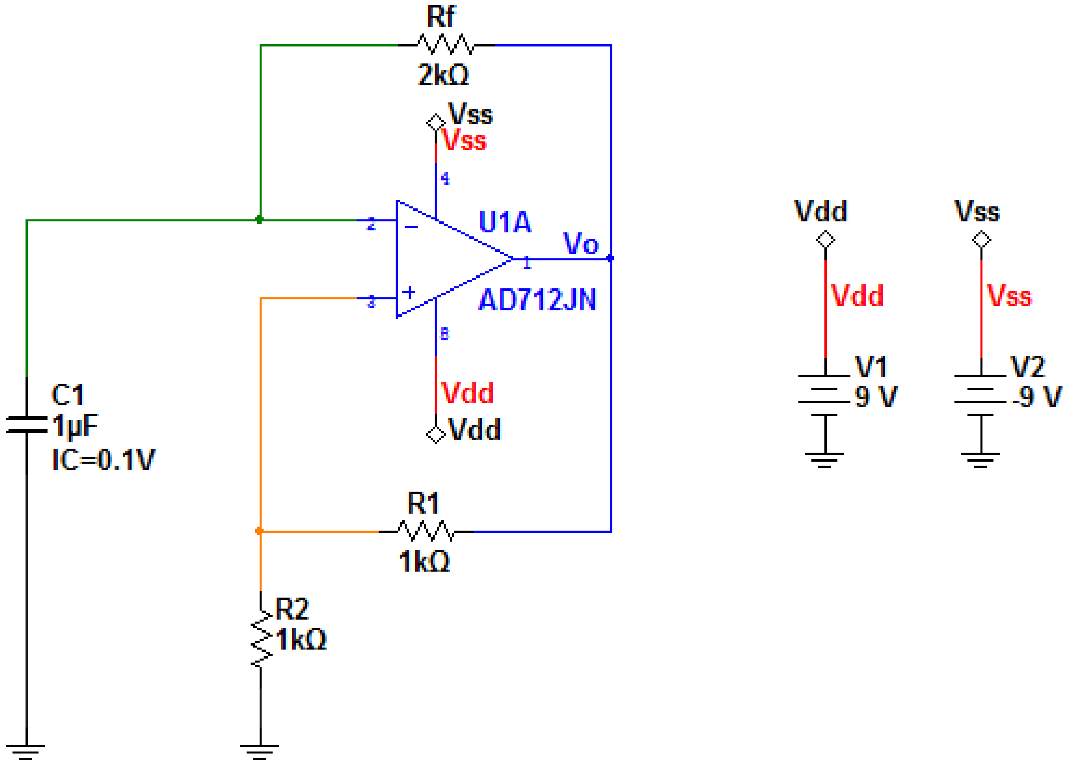
Recall from last lecture…

1. Oscillator vs. latch
2. Why the term relaxation oscillator

Questions?

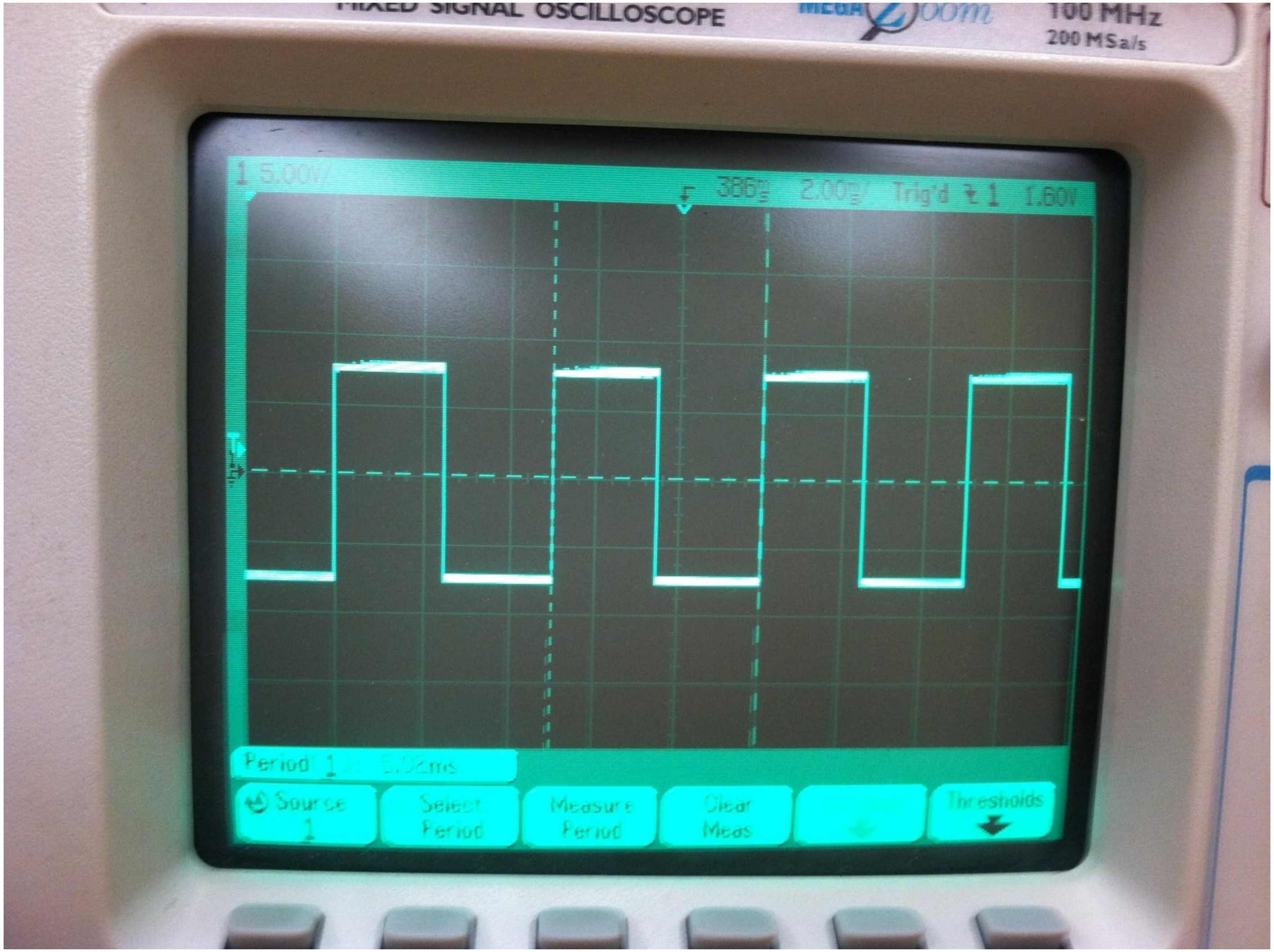
Goal of Lecture 2

Analytically determine the period and duty cycle in the circuit below.

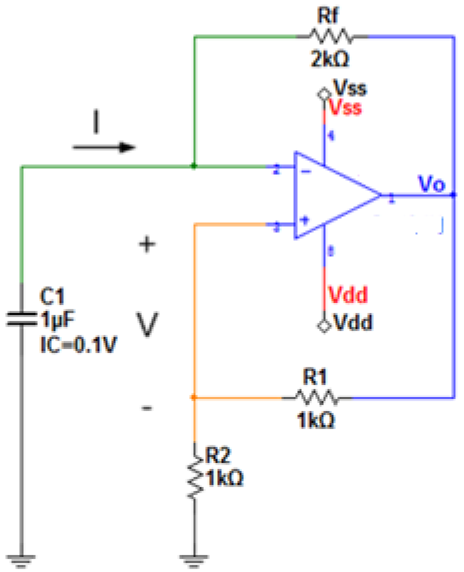


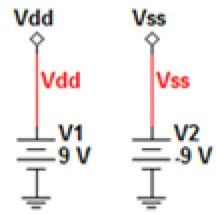
Recall : When you have questions, please ask!

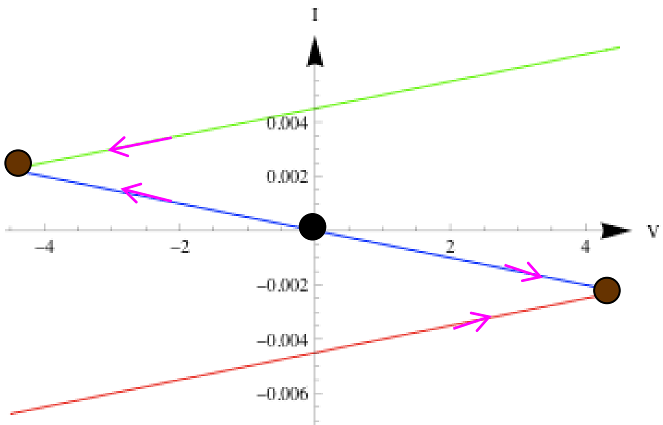
Definition of period and duty cycle



Finding the period







In the saturation region(s) :

$$v(t) = v_f + (v_i - v_f)e^{-t/R_f C}$$

In the positive saturation region :

$$v(t) = 9 + (-4.5 - 9)e^{-t/R_f C}$$

The period is 2x the switching time t_S :

$$4.5 = 9 - 13.5e^{-\left(t_S/R_f C\right)}$$

$$\text{Period} = 2t_S = 2\ln(3)R_f C$$

Finding the duty cycle

1. Recall (from last lecture): $v_p = \frac{R_2}{R_1 + R_2} v_0$

2. Circuit switches state when (positive saturation) : $v_p > v_n$

3. In other words : $V < \frac{R_2}{R_1 + R_2} \mathrm{V}_{dd}$

4. Hence, duty cycle is controlled by V_{dd} and V_{ss}

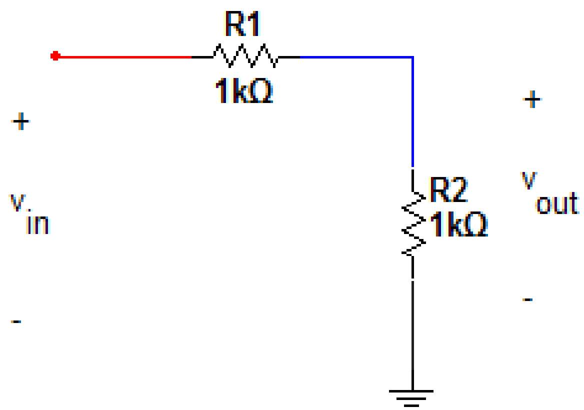
Now we will talk about chaos ☺

Note : This is NOT research!

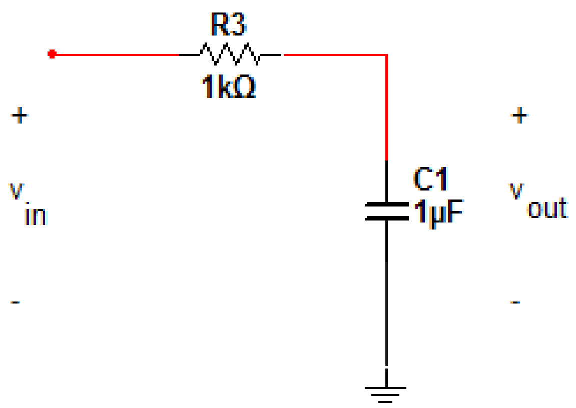
Introduction : Static vs. Dynamical Systems

1. Mathematical definition of a system

$$y(t) = S(x)(t) \quad y, x : \mathbb{R} \rightarrow \mathbb{R}, t \in \mathbb{R}$$



$$v_{out}(t) = \frac{R2}{R1+R2} v_{in}(t)$$



$$R3C1 \frac{dv_{out}}{dt} + v_{out} = v_{in}$$

$$v_{out}(t) = v_{out}(0) e^{-t/(R3C1)} + \frac{1}{R3C1} \int_0^t e^{\frac{-(t-\tau)}{R3C1}} v_{in}(\tau) d\tau$$

2. Concept of a linear time-invariant system

3. Various system behaviors : stable, unstable

Steady State Solutions of Differential Equations

Simple Harmonic Oscillator:

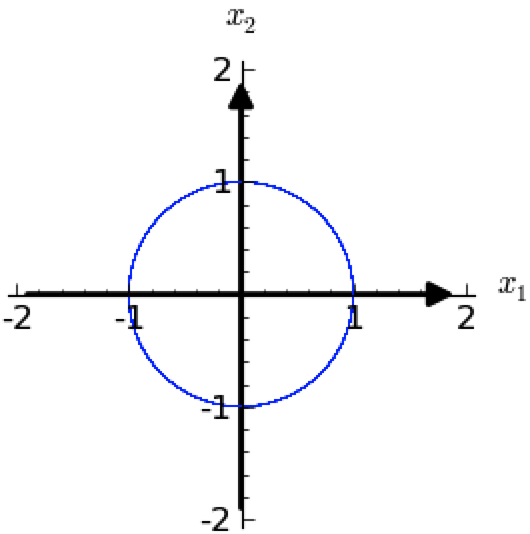
$$\ddot{x} + x = 0$$

$$\downarrow \; (x_1 \triangleq x, x_2 = \dot{x})$$

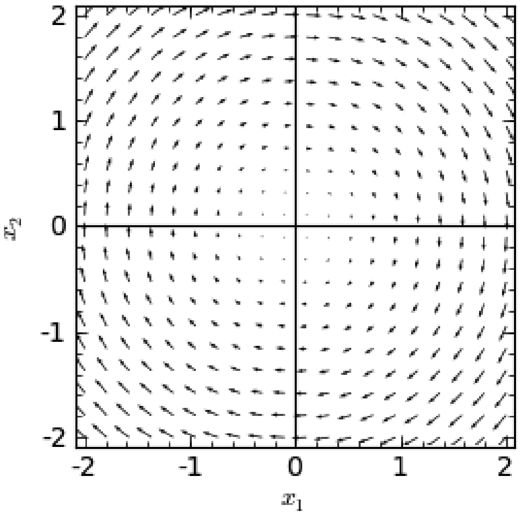
$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1$$

Phase portrait:



Vector field:



Plots were obtained using SAGE:
http://www.sagemath.org/index.html

Steady State Solutions of Differential Equations (contd.)

After stable, unstable and oscillatory behavior, we have quasi-periodicity [6]

$$\ddot{x} - (\lambda + z + x^2 - \frac{1}{2}x^4)\dot{x} + \omega_0{}^2 x = 0$$

$$\dot{z} = \mu - x^2$$

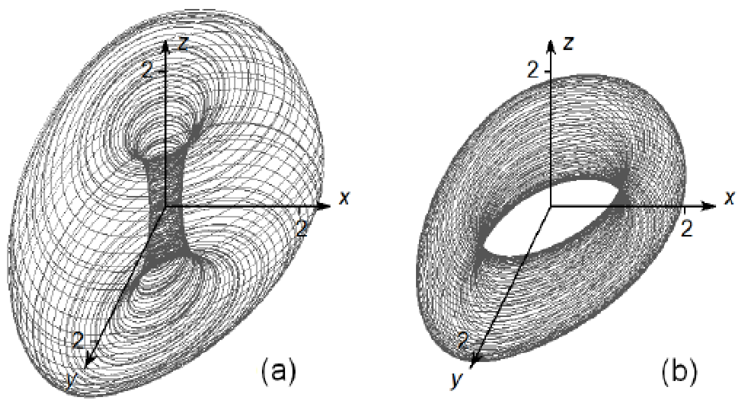


Figure 2. Portraits of attractor in the three-dimensional phase space of variables $(x,\ y = \dot{x}/\omega_0,\ z)$ for the model (1) with λ=0, $\omega_0 = 2\pi$ at μ=0.5 (a) and μ=0.9 (b)

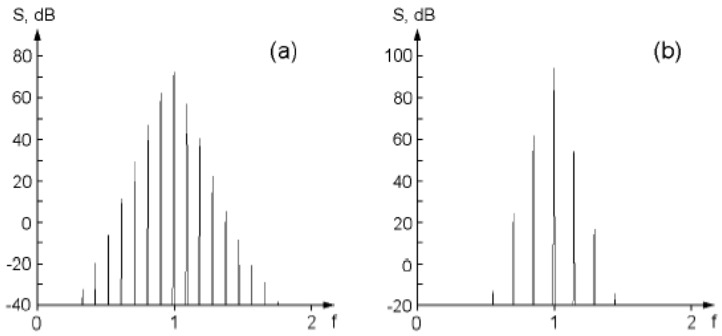


Figure 3. Fourier spectra of oscillations of the variable x on the attractor for the model (1) with λ=0, $\omega_0 = 2\pi$ at μ=0.5 (a) and μ=0.9 (b)

Steady State Solutions of Differential Equations - Chaotic Systems [1] [5] [10]

- "Birth" of Chaos: Lorenz Attractor [8]
  - Edward Lorenz introduced the following nonlinear system of differential equations as a crude model of weather in 1963:

$$\dot{x} = -\sigma \cdot x + \sigma \cdot y$$

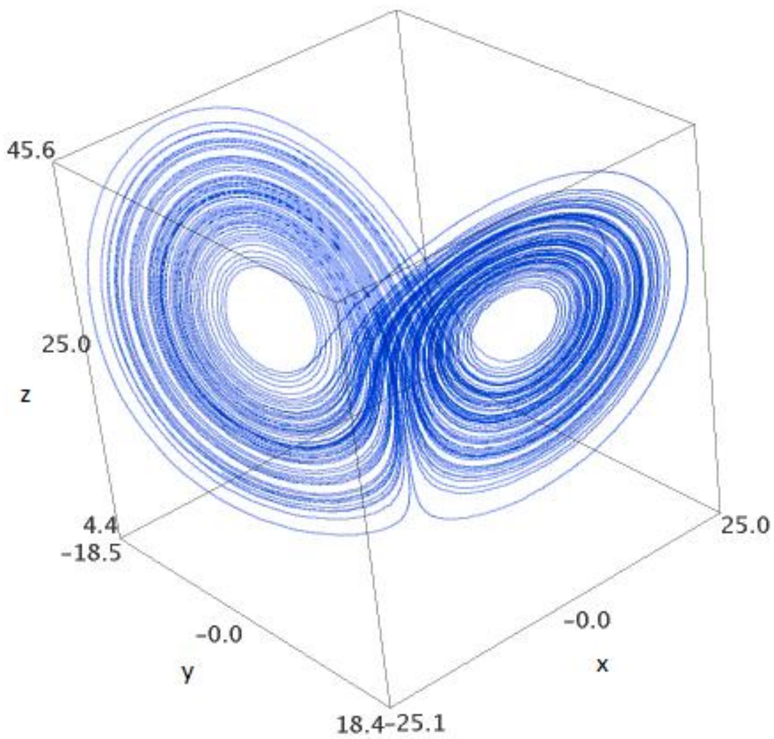
$$\dot{y} = \rho \cdot x - y - x \cdot z$$

$$\dot{z} = -\beta \cdot z + x \cdot y$$

Parameters: $\sigma = 10, \rho = 28, \beta = \frac{8}{3}$

ICs : $x_0 = 10, y_0 = 20, z_0 = 30,$

Simulation time: 100 seconds

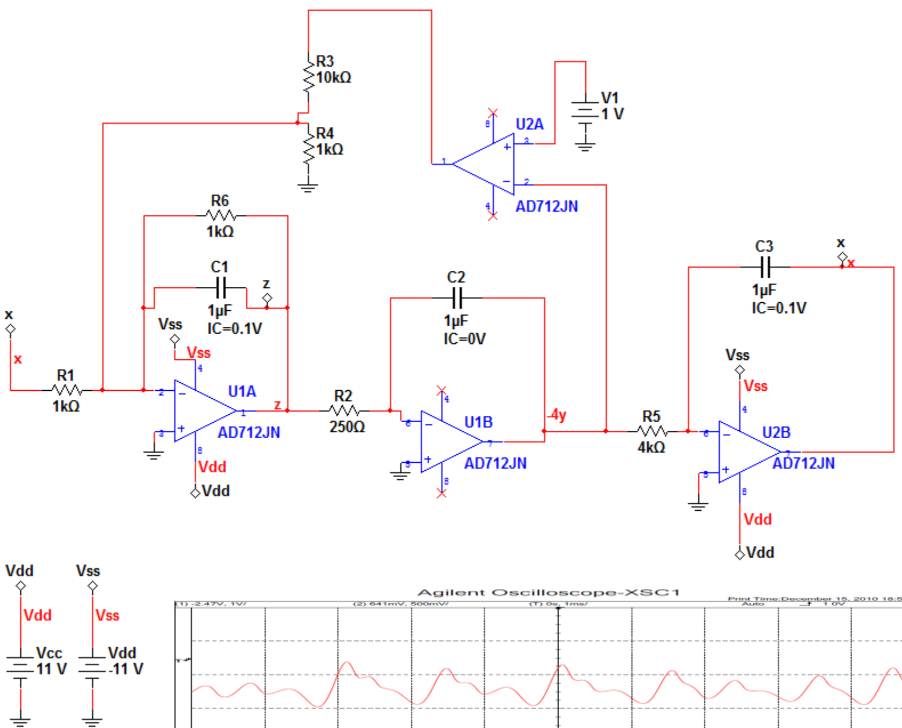


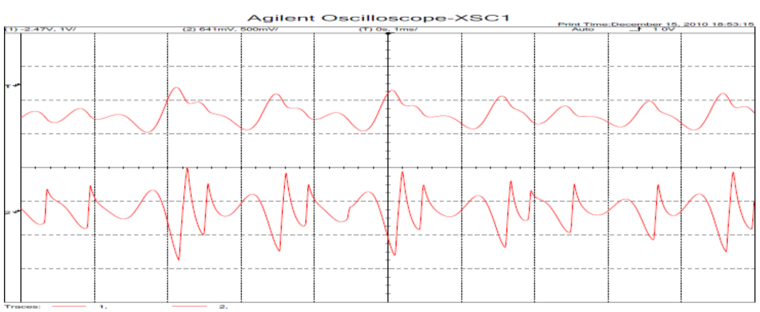
  - Lorenz discovered that model dynamics were extremely sensitive to initial conditions and the trajectories were aperiodic but bounded.
  - But, does chaos exist *physically*? Answer is: YES.

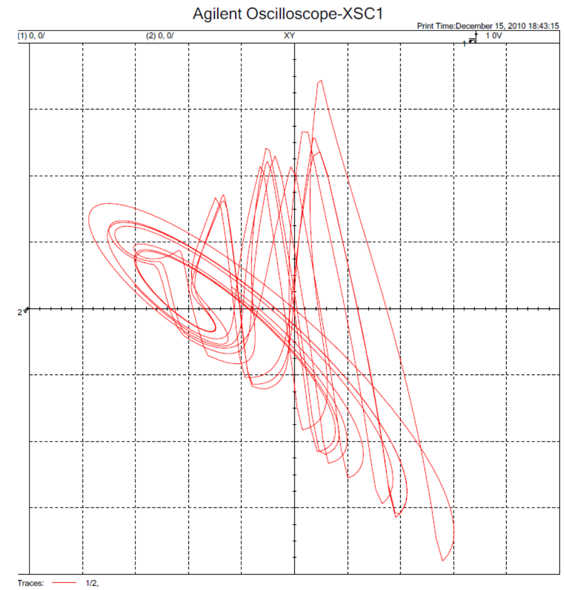
Physical Chaos - Sprott Circuits

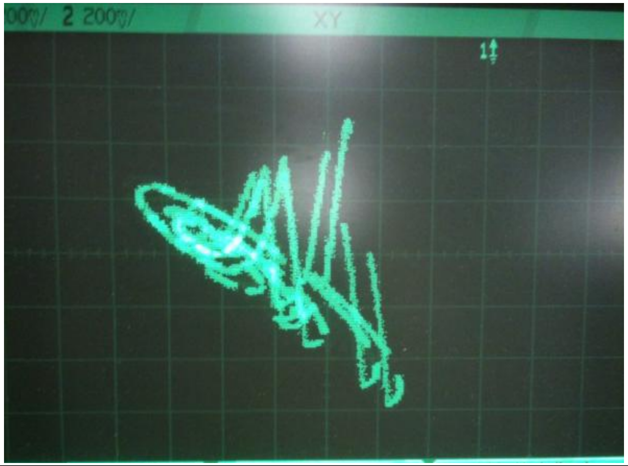
Simple Chaotic Circuit using Jerky Dynamics [9]

$$\dddot{x} + \ddot{x} + x + f(\dot{x}) = 0$$





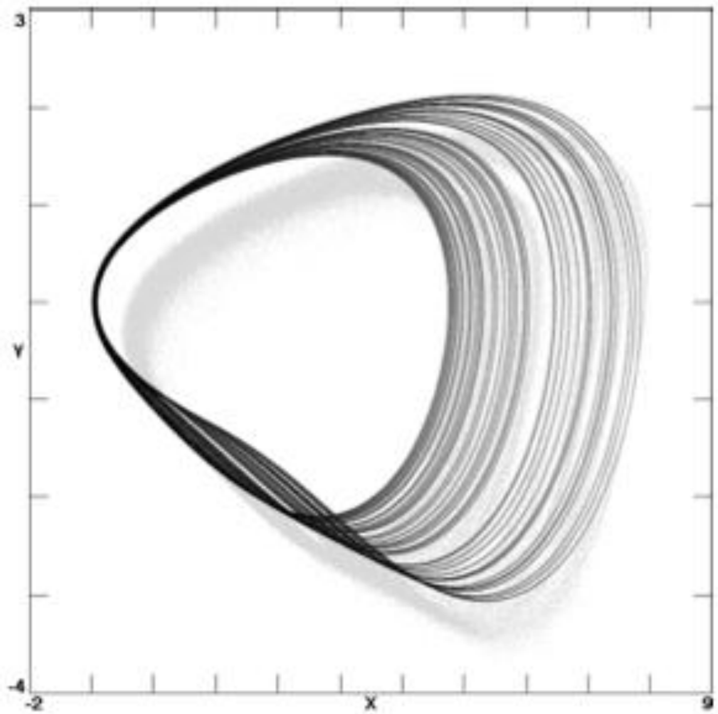




Some Properties of Chaotic Systems - "Dimension" of a Chaotic Attractor [9]

$$\dddot{x} + \alpha \ddot{x} + x + f(\dot{x}) = 0 \quad (9)$$

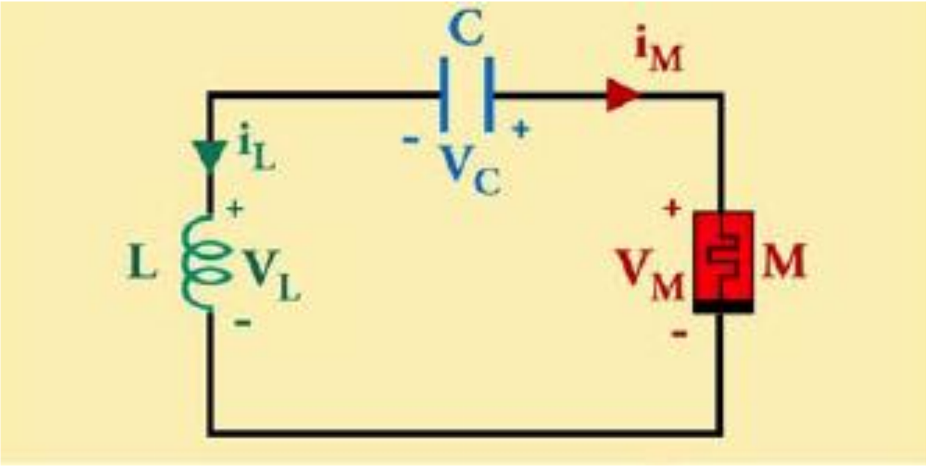
$$\alpha = 2.02,\ f(\dot{x}) = -\dot{x}^2, \text{i.c=(5,2,0)}$$

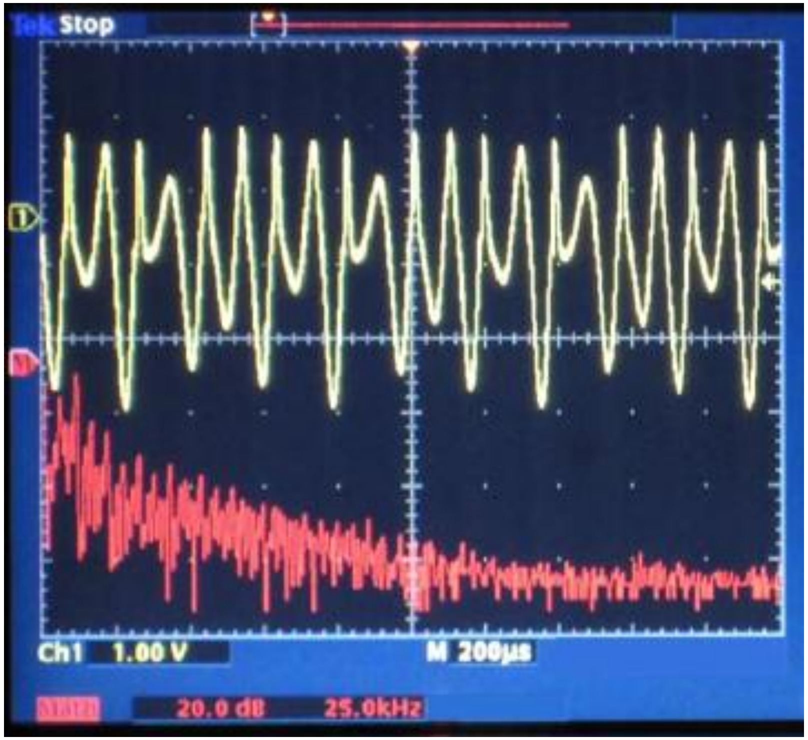


$$\lambda\text{=(0.0486,0,-2.0686)} \quad (10)$$

$$D_{KY} = 2 + \frac{0.0486 + 0}{|-2.0686|} \approx 2.02349$$

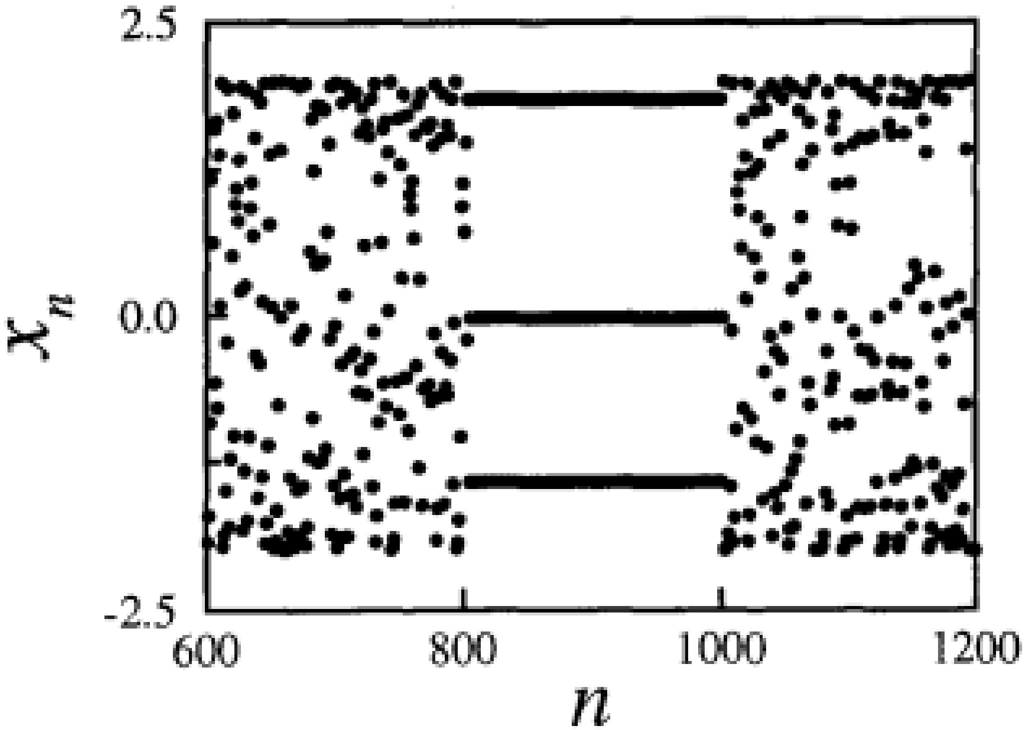
Some Properties of Chaotic Systems - The Frequency Spectrum [7]





An Application of Chaos: Human arrhythmia control [4]

1. Arrhythmia = “not in rhythm” = bad



Conclusions

What we have learned:

1. Separate dynamical from non-dynamical component (if possible) when analyzing circuits
2. Nonlinearity is essential for practical oscillators
3. Glimpse of chaos

Questions??

Food for thought : WHAT DOES “nonlinear” mean?

References

1. Alligood, K. T., Sauer, T. and Yorke, J. A. *Chaos: An Introduction to Dynamical Systems*. Springer, 1997.
2. Chua, L. O. "Memristor-The Missing Circuit Element". *IEEE Transactions on Circuit Theory*, Vol. CT-18, No. 5, pp. 507- 519. September 1971.
3. Chua, L. O. and Kang, S. M. "Memristive Devices and Systems". *Proceedings of the IEEE*, Vol. 64, No. 2, pp. 209- 223. February 1976.
4. Christini et. al. "Nonlinear-dynamical arrhythmia control in humans". *Proceedings of the National Academy of Sciences of the United States of America.* Vol. 98 , No. 10, pp. 5827–5832, 2001.
5. Hirsch, M. W., Smale S. and Devaney, R. *Differential Equations, Dynamical Systems and An Introduction To Chaos.* 2nd Edition, Elsevier, 2004.
6. Kuznetsov, A. P. et. al. "A Simple Autonomous Quasiperiodic Self-Oscillator". Preprint submitted to Elsevier Science, June 30th 2009
7. Muthuswamy, B. and Chua, L. O. "Simplest Chaotic Circuit". *International Journal of Bifurcation and Chaos*, vol. 20, No. 5, pp. 1567-1580. May 2010.
8. Lorenz, E. N. "Deterministic Nonperiodic Flow". *Journal of Atmospheric Sciences*, vol. 20, pp/ 130-141, 1963.
9. Sprott, J. C. "Elegant Chaos". World Scientific, pp. 758-763, 2010.
10. Strogatz, S. H. *Nonlinear Dynamics and Chaos*. Addison-Wesley, 1994.