Chaos is Fun!

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Session : Teaching or Learning Advanced Mathematics

Saturday, August 3rd 2012

Abstract : In this paper, we will discuss chaotic dynamics. Commonly associated with the "Butterfly Effect", we will see chaotic dynamics can be used to gain insights into varied concepts such as steady state solutions of differential equations, dimensions of phase space objects and the Fourier transform. All code in this paper uses the open source SAGE (Software for Algebraic and Geometric Experimentation) tool.

Bharathwaj "Bart" Muthuswamy

muthuswamy@msoe.edu

Assistant Professor of Electrical Engineeirng

Milwaukee School of Engineering (MSOE)

BS (2002), MS (2005), PhD (2009) from Cal

Advisor: Dr. Leon O Chua, co-advisor: Dr. Pravin Varaiya

http://www.harpgroup.org/muthuswamy

Outline

I. Prerequisites for understanding this talk:
  1. First course in circuit theory*
  2. First course in differential equations

II. Introduction
  1. Fundamental Circuit Theory [2] [3]
  2. Static vs. Dynamical systems

III. Steady-state Solutions of Differential equations
  1. Simple Harmonic Oscillator
  2. Quasi-periodicity
  3. Chaos [1] [5] [10]

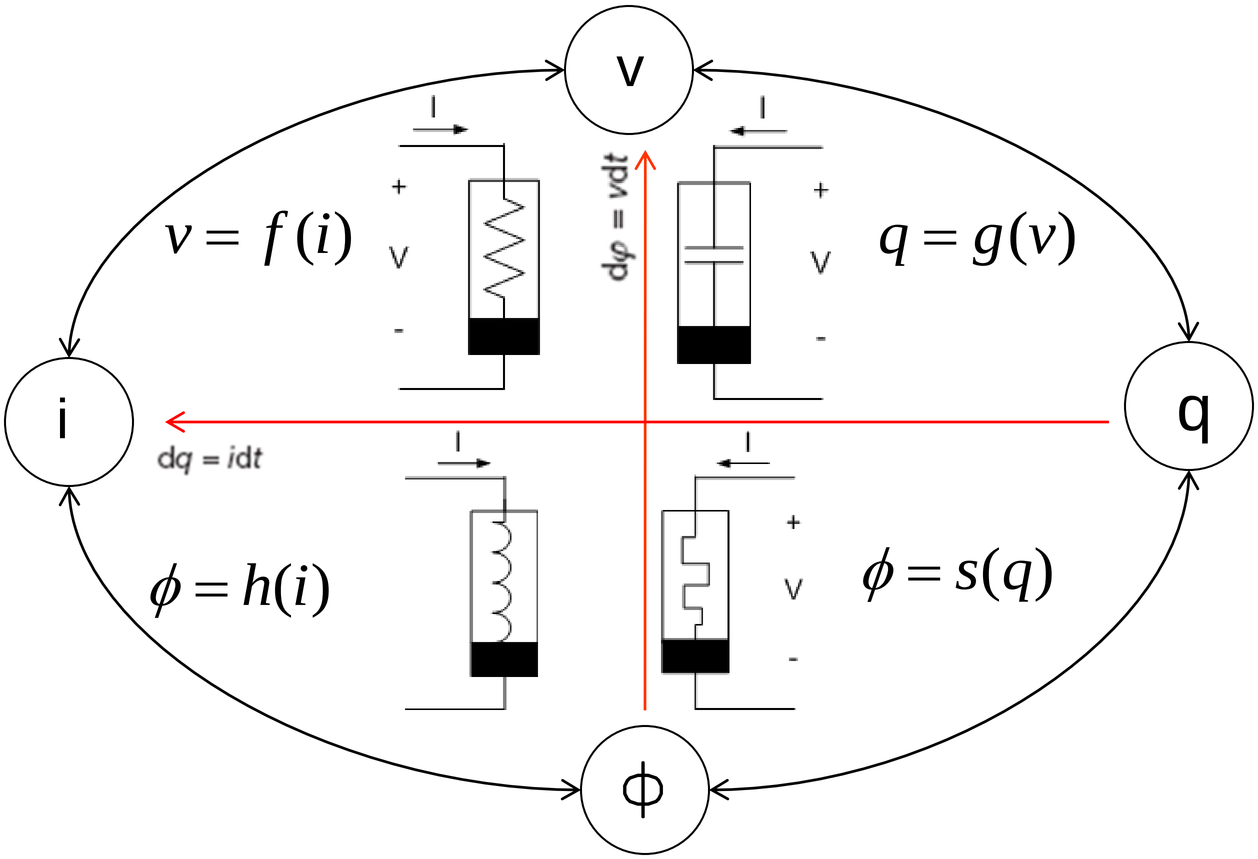
IV. Some Properties of Chaotic Systems
  1. The "Dimension" of a chaotic attractor [9]
  2. The Frequency Spectrum [7]

V. Physical Chaos – Sprott circuits

VI. An Application of Chaos : Human arrhythmia control [4]

VII. References

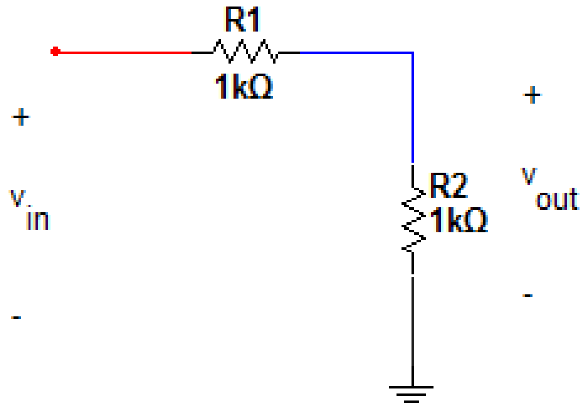
Introduction : Fundamental Circuit Theory [2] [3]



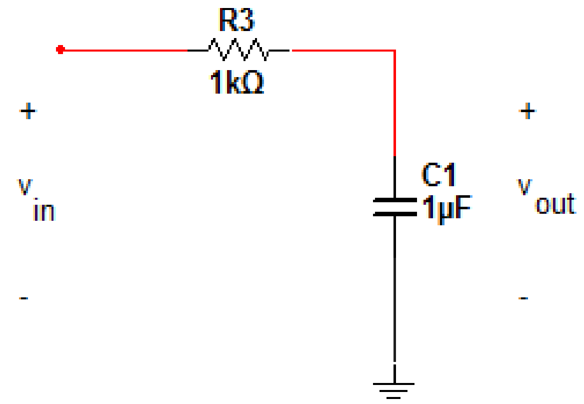
Introduction :
Static vs. Dynamical Systems

1. Mathematical definition of a system

$$y(t) = S(x)(t) \quad y, x : \mathbb{R} \to \mathbb{R}, t \in \mathbb{R} \qquad (1)$$



$$v_{out}(t) = \frac{R2}{R1+R2} v_{in}(t) \qquad (2)$$



$$R3C1 \frac{dv_{out}}{dt} + v_{out} = v_{in} \qquad (3)$$

$$v_{out}(t) = v_{out}(0)e^{-t/(R3C1)} + \frac{1}{R3C1}\int_0^t e^{\frac{-(t-\tau)}{R3C1}} v_{in}(\tau)d\tau \qquad (4)$$

2. Concept of a linear time-invariant system

3. Various system behaviors : stable, unstable

Steady State Solutions of Differential Equations

Simple Harmonic Oscillator:

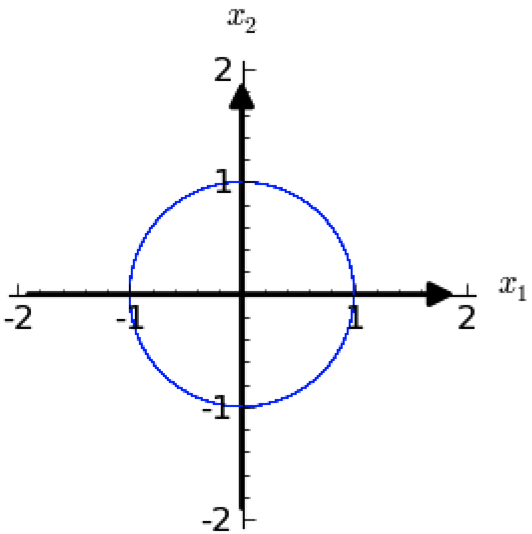
$$\ddot{x} + x = 0 \qquad (5)$$

$$\downarrow \ (x_1 \triangleq x, x_2 = \dot{x})$$

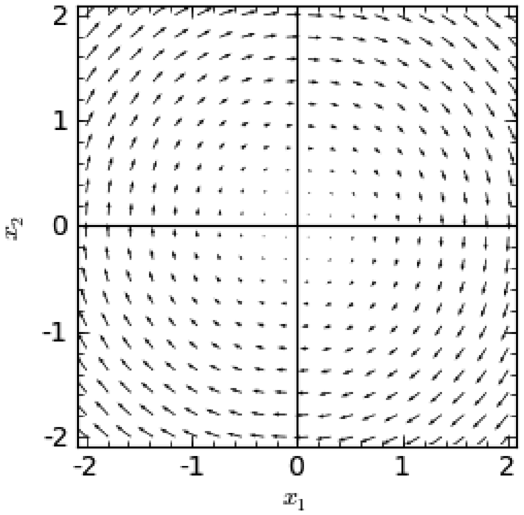
$$\dot{x}_1 = x_2 \qquad (6)$$

$$\dot{x}_2 = -x_1$$

Phase portrait:



Vector field:



Plots were obtained using SAGE:
http://www.sagemath.org/index.html

Steady State Solutions of Differential Equations (contd.)

After stable, unstable and oscillatory behavior, we have quasi-periodicity [6]

$$\ddot{x} - (\lambda + z + x^2 - \frac{1}{2}x^4)\dot{x} + \omega_0{}^2 x = 0 \quad (7)$$

$$\dot{z} = \mu - x^2$$

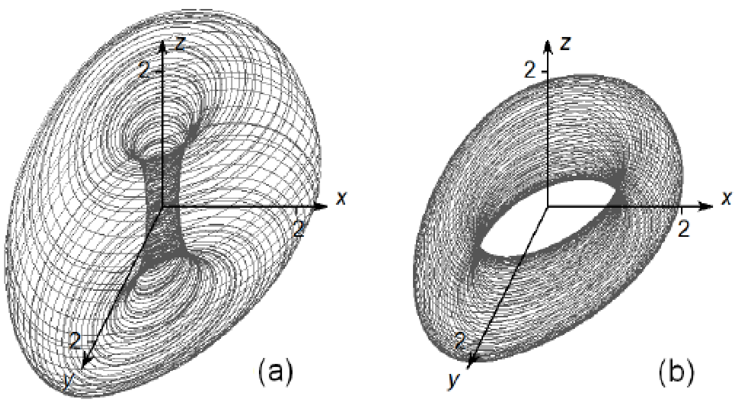


Figure 2. Portraits of attractor in the three-dimensional phase space of variables $(x,\ y = \dot{x}/\omega_0,\ z)$ for the model (1) with λ=0, $\omega_0 = 2\pi$ at μ=0.5 (a) and μ=0.9 (b)

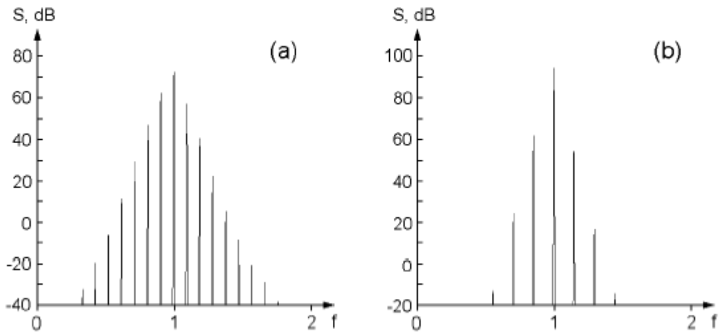


Figure 3. Fourier spectra of oscillations of the variable x on the attractor for the model (1) with λ=0, $\omega_0 = 2\pi$ at μ=0.5 (a) and μ=0.9 (b)

Steady State Solutions of Differential Equations - Chaotic Systems [1] [5] [10]

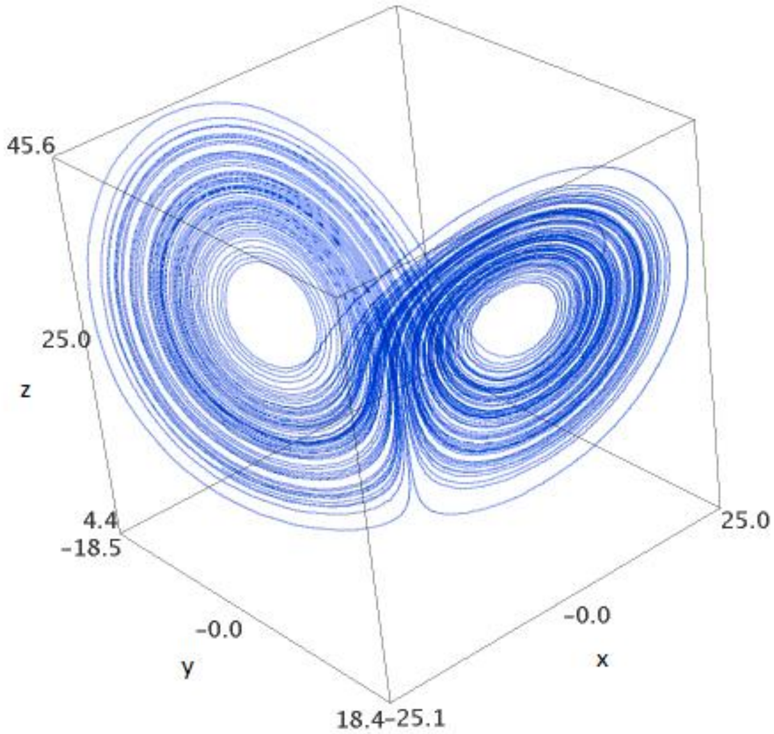
- "Birth" of Chaos: Lorenz Attractor [8]
  - Edward Lorenz introduced the following nonlinear system of differential equations as a crude model of weather in 1963:

$$\dot{x} = -\sigma \cdot x + \sigma \cdot y$$
$$\dot{y} = \rho \cdot x - y - x \cdot z \qquad (8)$$
$$\dot{z} = -\beta \cdot z + x \cdot y$$

Parameters: $\sigma = 10, \rho = 28, \beta = \frac{8}{3}$

ICs: $x_0 = 10, y_0 = 20, z_0 = 30,$

Simulation time: 100 seconds

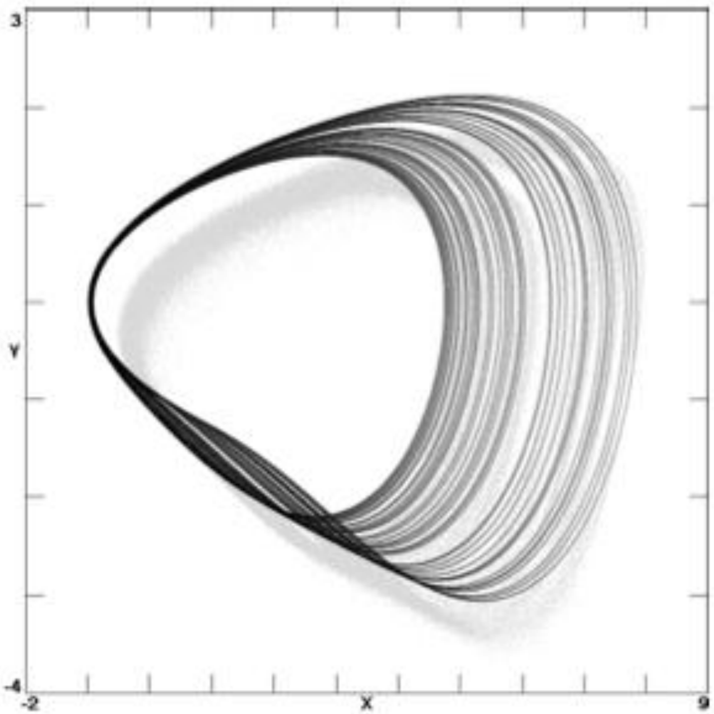


  - Lorenz discovered that model dynamics were extremely sensitive to initial conditions and the trajectories were aperiodic but bounded.
  - But, does chaos exist *physically*? Answer is: YES.

Some Properties of Chaotic Systems - "Dimension" of a Chaotic Attractor [9]

$$\dddot{x} + \alpha \ddot{x} + x + f(\dot{x}) = 0 \quad (9)$$

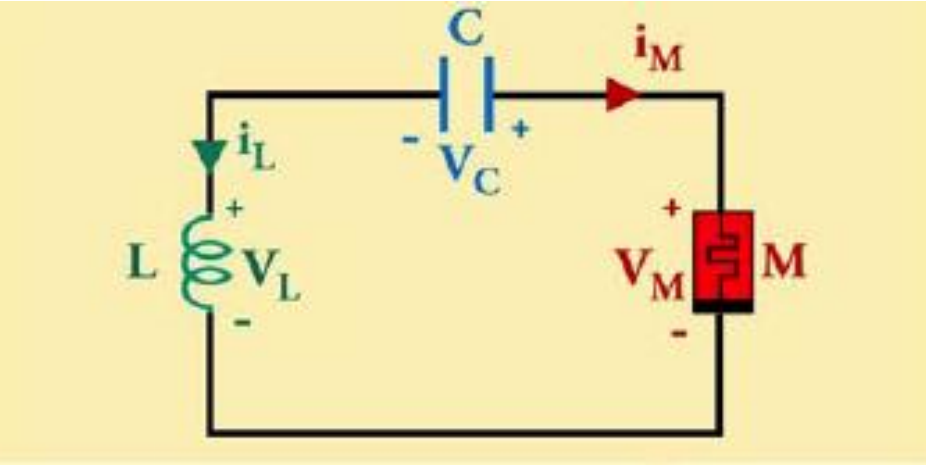
$$\alpha = 2.02,\ f(\dot{x}) = -\dot{x}^2, \text{i.c=(5,2,0)}$$

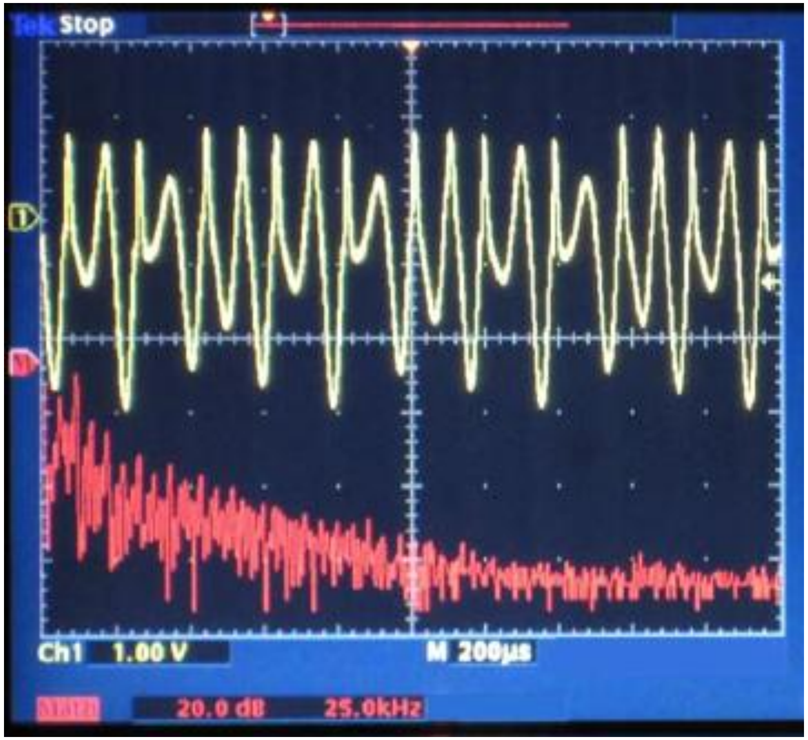


$$\lambda\text{=(0.0486,0,-2.0686)} \quad (10)$$

$$D_{KY} = 2 + \frac{0.0486 + 0}{|-2.0686|} \approx 2.02349$$

Some Properties of Chaotic Systems - The Frequency Spectrum [7]

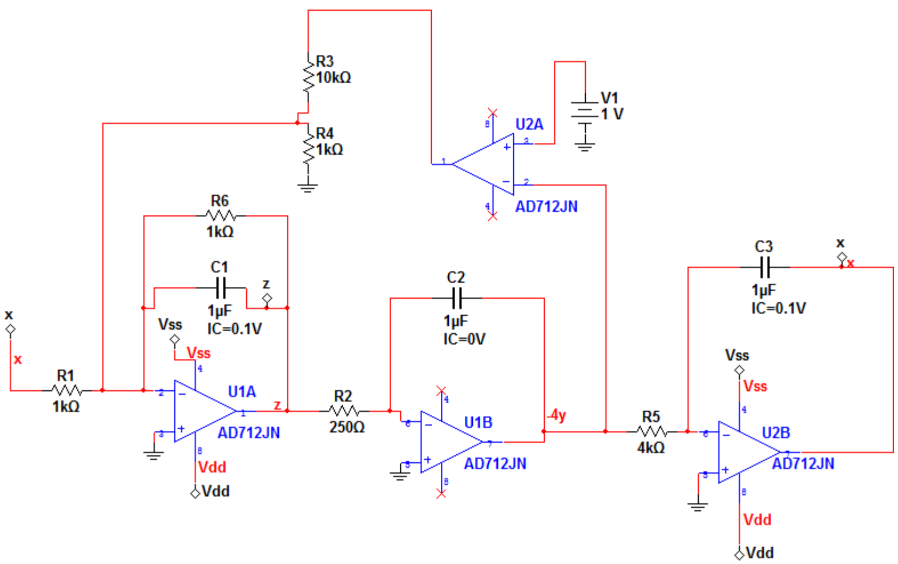


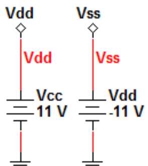


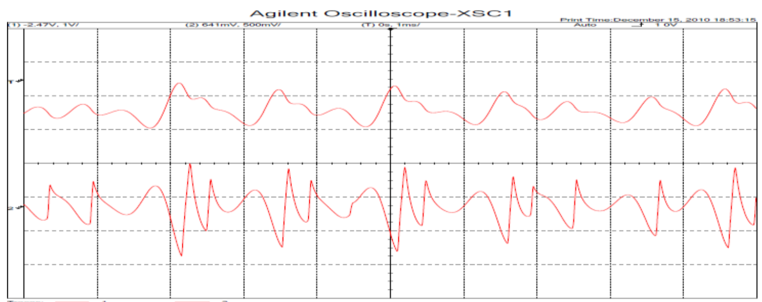
Physical Chaos - Sprott Circuits

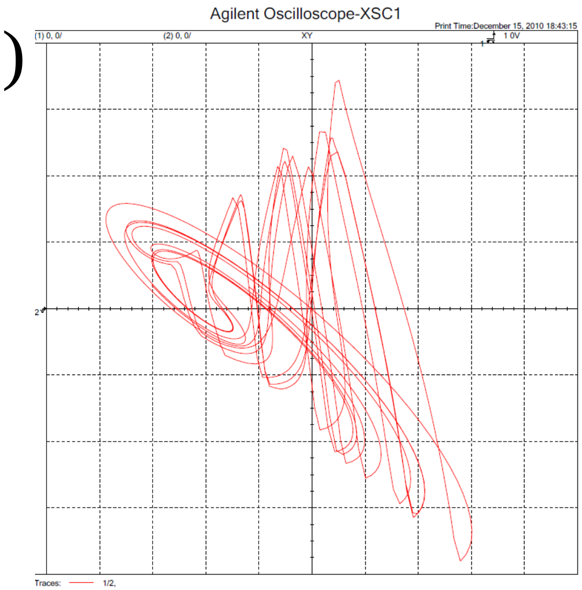
Simple Chaotic Circuit using Jerky Dynamics [9]

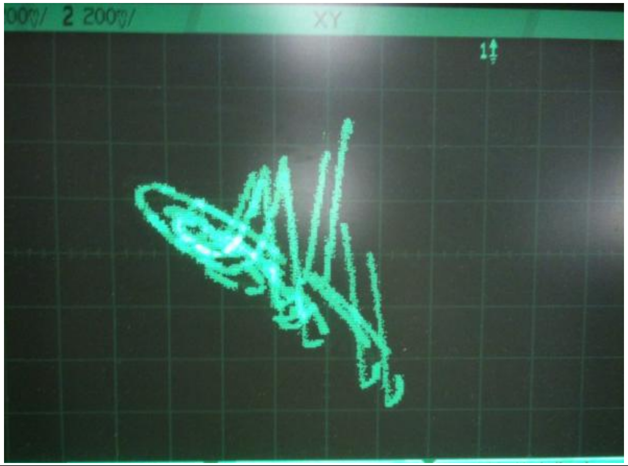
$$\dddot{x} + \ddot{x} + x + f(\dot{x}) = 0 \quad (11)$$





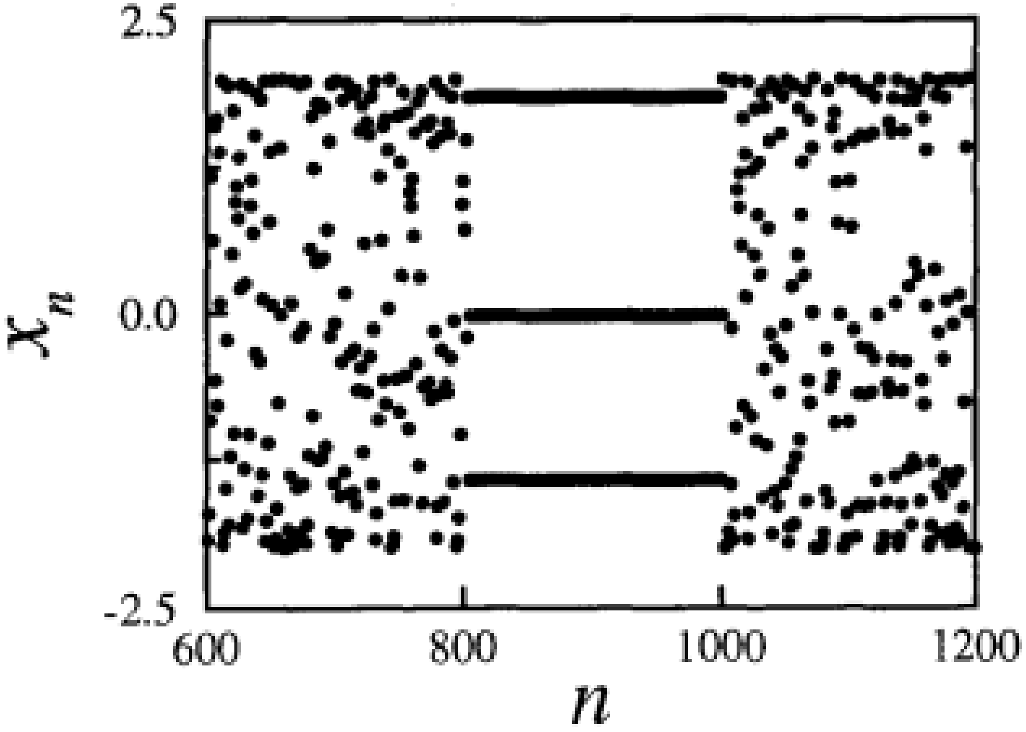






An Application of Chaos: Human arrhythmia control [4]

1. Arrhythmia = “not in rhythm” = bad



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Questions?